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ERROR ESTIMATES FOR THE APPROXIMATION OF AN UNKNOWN CONSTANT CO--ETC(U)

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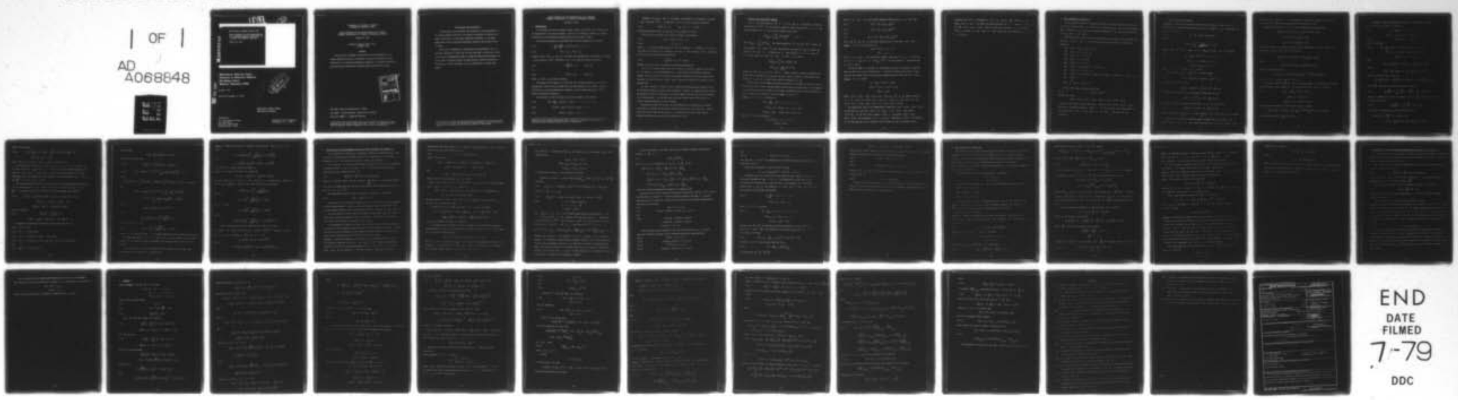

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ERROR ESTIMATES FOR THE APPROXIMATION OF AN UNKNOWN CONSTANT COEFFICIENT IN A PARTIAL DIFFERENTIAL EQUATION

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See back page for 1473

December 1978

(Received September 11, 1978)

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

A priori error estimates are derived for the approximation of an inverse problem in which it is desired to identify an unknown constant coefficient in a one space dimensional parabolic or two space dimensional elliptic partial differential equation whose general form is known.

AMS (MOS) Subject Classification - 65M30

Key Words - Inverse problem, identification problem

Work Unit Number 7 - Numerical Analysis

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS78-02737.

SIGNIFICANCE AND EXPLANATION

In this paper error estimates are derived for the approximation of an inverse problem in which it is desired to determine an unknown constant coefficient in a one space dimensional parabolic or two space dimensional elliptic partial differential equation whose general form is known.

This type of problem has a wide variety of applications (e.g. in pollution problems in rivers and oil flow problems) in which one is trying to build a mathematical model to study the underlying phenomena, but is not able to measure directly certain material properties which are necessary to forming the model (e.g. the diffusivity in a heat flow model).

ERROR ESTIMATES FOR THE APPROXIMATION OF AN UNKNOWN
CONSTANT COEFFICIENT IN A PARTIAL DIFFERENTIAL EQUATION

Richard S. Falk

1. INTRODUCTION.

In this paper we wish to present further results on a priori error estimates for the approximation of inverse problems in which it is desired to identify an unknown coefficient in a differential equation whose general form is known.

In two previous papers we considered the two point boundary value problem:

$$-\frac{d}{dx}\left(a\,\frac{du^a}{dx}\right) + c(x)u^a = f(x), \quad 0 < x < 1 \tag{1.1}$$

$$u^a(0) = g_1, \quad u^a(1) = g_2 . \tag{1.2}$$

Assuming $f(x)$, $c(x)$, g_1, and g_2 are known the problems were to determine a constant a and a function $u^a(x)$ satisfying (1.1), (1.2) and the auxiliary condition

$$-a\,\frac{du^a}{dx}(0) = g_3 \qquad \text{(see [3])} \tag{1.3}$$

or

$$u^a(x^*) = g_4 \qquad \text{(see [4])}, \tag{1.4}$$

where g_3 and g_4 are given constants.

The purpose of this paper is to show how the techniques employed in [3] can be generalized to derive error estimates for the approximation of an unknown constant coefficient in one space-dimensional parabolic and two space-dimensional elliptic partial differential equations.

In particular, we will consider the initial boundary value problem

$$u^a_t - au^a_{xx} + c(x,t)u^a = f(x,t), \quad 0 < x < 1, \quad t > 0 \tag{1.5}$$

$$u^a(0,t) = g_5(t), \; u^a(1,t) = g_6(t), \quad t > 0 \tag{1.6}$$

$$u^a(x,0) = g_7(x), \quad 0 < x < 1 \tag{1.7}$$

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Assuming f, c, g_5, g_6, and g_7 are known, the problem is to determine a constant a and a function $u^a(x,t)$ satisfying (1.5)-(1.7) and the auxiliary condition:

$$(1.8) \qquad -au_x^a(0,t^*) = g_8 \qquad (g_8 \text{ and } t^* > 0 \text{ given}) .$$

We shall henceforth denote by Problem (P) the problem (1.5)-(1.8).

We will also investigate the elliptic boundary value problem

$$(1.9) \qquad -a\Delta u^a + c(x)u^a = f(x), \quad x \in \Omega$$

$$(1.10) \qquad u^a = g \quad \text{on } \Gamma$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ with boundary Γ. Assuming f, c, and g are known, the problem is to determine a constant a and a function $u^a(x)$ satisfying (1.9)-(1.10) and the auxiliarly condition:

$$(1.11) \qquad a\frac{\partial u^a}{\partial n}(x^*) = g^* \quad (g^* \text{ given}) .$$

Problem (1.9)-(1.11) will henceforth be denoted by Problem (E).

As in [3], our main concern will be to derive an a priori estimate for the error $|a - \alpha|$ where α is an approximation to a obtained by an appropriate numerical scheme. To do so we will first need to determine conditions on the data under which Problems (P) and (E) are well posed. These results are contained in Sections 3 and 5 respectively.

We then consider in Sections 4 and 6 respectively approximation schemes for the two problems and derive a priori error estimates. Finally for completeness, we include an appendix in which we give proofs of several lemmas which we need in the derivation of our main results. For the most part they only involve minor modifications of results already appearing in the literature.

References to much of the work on the determination of parameters in elliptic and parabolic problems can be found in the bibliographies of [10] and [11]. In particular we wish to mention the work of J. R. Cannon, since some of the results obtained and techniques used here are extensions of [2].

2. NOTATION AND PRELIMINARY RESULTS.

For Ω a bounded domain in $\mathbb{R}^2$ or $I = [0,1]$ and k a nonnegative integer we shall denote by $W^{k,p}$ the usual Sobolev spaces of function defined on Ω or I (the intended domain will be clear from the context) with norms

$$\|u\|_{W^{k,p}} = \left(\sum_{|j|=0}^{k} \|D^j u\|_{L_p}^p\right)^{1/p} \qquad 1 \le p < \infty$$

and $\|u\|_{W^{k,\infty}} = \sum_{|j|=0}^{k} \|D^j u\|_{L_\infty}$. We further denote by H^k the space $W^{k,2}$ and by H_0^1 the closure of C_0^∞ in the H^1 norm. We will use the notation $\|u\|_k$ to denote the norms in H^k, and $(\cdot,\cdot)$ to denote the L_2 inner product in Ω or I. Also for X a normed space with norm $\|\cdot\|_X$ and $u : [0,T] \to X$, we define

$$\|u\|^2_{L_2(X)} = \int_0^T \|u(\cdot,t)\|_X^2 dt \quad \text{and}$$

$$\|u\|_{L_\infty(X)} = \sup_{0 \le t \le T} \|u(\cdot,t)\|_X .$$

Finally we shall let $I_T = [0,1] \times [0,T]$, C denote a generic constant depending only on the data of Problem (P) or (E), and $C(b_*,b^*)$ denote a generic constant which may depend on b_* and b^* as well as the data of Problem (P) or (E).

We now state several lemmas which will be needed in the proofs of our main results. The first two are results from perturbation theory and we provide proofs in the appendix for the special cases we consider in this paper, although results of this general kind can be found in the literature (see for example [5] and [7]).

Lemma 1: Let z^a be the solution of

$$z_t^a - az_{xx}^a + cz^a = F, \quad 0 < x < 1, \quad t > 0$$

$$z^a(0,t) = 0, \quad z^a(1,t) = 0, \quad t > 0, \quad \text{and}$$

$$z^a(x,0) = G(x) \qquad (\text{with } G(0) = G(1) = 0) ,$$

and let z^0 be the solution of

$$z_t^0 + cz^0 = F, \quad 0 < x < 1, \quad t > 0$$
$$z^0(x,0) = G(x) .$$

Then if $F,c,$ and G are sufficiently smooth we have for all $0 < a < 1/4$ that

(2.1) $$\| [z^a - z^0](\cdot,t) \|_0 \leq K_4^t a^{1/4} ,$$

(2.2) $$\| [z_t^a - z_t^0](\cdot,t) \|_0 \leq K_5^t a^{1/4} ,$$

and

(2.3) $$\| [F - cz^a - z_t^a](\cdot,t) \|_0 \leq K_6^t a^{1/4}$$

where K_4^t, K_5^t, and K_6^t are constants depending only on the data c,F,G and t.

Lemma 2: Let z^a be the solution of

$$- a\Delta z^a + cz^a = f - c\omega \quad \text{in } \Omega$$

$$z^a = 0 \quad \text{on } \Gamma .$$

Then if f,c,ω and Γ are sufficiently smooth, there exists $0 < \varepsilon_0 < 1$ such that for all $0 < a \leq \varepsilon_0^2$, $\|f - c\omega - cz^a\|_{L_4} \leq Ca^{1/8}$ for some constant C depending only on $f,c,\omega,$ and Ω.

The third lemma is a statement of the maximum principle for parabolic equations which we shall use in the following form. (A general reference for results of this type and for similar results for elliptic problems which we shall also use is [12]).

Lemma 3: Let v satisfy

$$v_t - av_{xx} + c(x,t)v = F(x,t)$$

$$v(0,t) = G_1(t), \quad v(1,t) = G_2(t)$$

$$v(x,0) = G_3(x)$$

where $G_1(0) = G_3(0)$, $G_2(0) = G_3(1)$ and $c,F,G_1,G_2,$ and G_3 are smooth functions. If $a > 0$, $c(x,t) \geq 0$, $F(x,t) \leq 0$, $G_1(t) \leq 0$, $G_2(t) \leq 0$, and $G_3(x) \leq 0$, for $(x,t) \in (0,1) \times (0,T]$, then $v(x,t) \leq 0$, $(x,t) \in [0,1] \times [0,T]$.

We remark that even if the compatibility conditions $G_1(0) = G_3(0)$, $G_2(0) = G_3(1)$ do not hold, we still have $v(x,t) \leq 0$, $(x,t) \in (0,1) \times (0,T]$, even though $v \notin C^0[I_T]$. To see this take a sequence $G_3^n(x) \leq 0$ satisfying $G_3^n(0) = G_2(0)$, $G_3^n(1) = G_2(0)$ and converging to G_3 in $L_2(0,1)$. Denoting by $v^n(x,t)$ the solution of the corresponding initial boundary value problem, we have by standard energy

estimates that $v^n(\cdot,t)$ converges to $v(\cdot,t)$ in $L_2(0,1)$. Now $v^n(x,t) \leq 0$ by lemma 3 and v^n and v are both continuous functions for $t > 0$. Hence if $v(x,t) > 0$ for some $(x,t) \in (0,1) \times (0,T]$, then we could find a function $\psi \geq 0$ such that $(v^n,\psi) \leq 0$ for all n, but $(v,\psi) > 0$. This contradicts the convergence of v^n to v in $L_2(0,1)$.

3. WELL POSEDNESS OF PROBLEM (P).

In this section we wish to determine some simple conditions under which problem (1.5)-(1.8) is well posed, i.e. there exists a unique positive solution to the equation $Q(a) \equiv -au_x^a(0,t^*) = g_8$ which is continuously dependent on g_8 (we shall assume that c, f, g_5, g_6 and g_7 are fixed).

Although theorems under alternative hypotheses are possible, we shall assume that the data c, f, g_5, g_6, and g_7 are sufficiently smooth and satisfy the following conditions:

(H0) $g_5(0) = g_7(0), \quad g_6(0) = g_7(1)$

(H1) $c(x,t) \geq 0, \quad (x,t) \in I_T$

(H2) $g_5'(t) \geq 0, \quad t \in [0,T]$

(H3) $g_6'(t) \leq 0, \quad t \in [0,T]$

(H4) $g_7''(x) \leq 0, \quad x \in [0,1]$

(H5) $f(x,t) - c(x,t)g_7(x) \leq 0, \quad (x,t) \in I_T$

(H6) $c_t(x,t) \leq 0, \quad (x,t) \in I_T$

(H7) $f_t(x,t) - c_t(x,t)g_7(x) - c_t(x,t)\{(1-x)[g_5(t) - g_5(0)]\} \leq 0, \quad (x,t) \in I_T$

(H8) $g_7'(0) \leq 0$

(H9) $\int_0^{t^*} g_5'(\tau)d\tau - g_7'(0) > 0$.

We note that these conditions imply that

$$g_5(t^*) > g_6(t^*) \ . \tag{3.1}$$

To see this, write

$$g_6(0) = g_7(1) = g_7(0) + g_7'(0) + g_7''(\xi) = g_5(0) + g_7'(0) + g_7''(\xi)$$

(using (H0) and a Taylor series expansion). Hence by (H4) and (H8), $g_6(0) \leq g_5(0)$ with strict inequality holding if $g_7'(0) < 0$. By (H2) and (H3) our desired result is true unless both $g_7'(0) = 0$ and $g_5'(t) = 0$, $t \in [0,t^*]$. (H9) rules out this possibility.

We now prove the following:

Theorem 1: Suppose that hypotheses (H0)-(H9) are satisfied. Then for any $g_8 > 0$, there exists a unique positive solution of $Q(a) \equiv -au_x^a(0,t^*) = g_8$. Furthermore, for all $0 < a,b \le a^*$,

(3.2) $$|a - b| \le |Q(a) - Q(b)|/K(a^*)$$

where

$$K(a^*) = \frac{1}{2}\int_0^{t^*} \frac{g_5'(\tau)}{\sqrt{\pi a^*(t-\tau)}}\,d\tau - g_7'(0).$$

Proof: Write $u^a = z^a + r$ where $r = (1-x)g_5(t) + xg_6(t)$ and z^a satisfies

(3.3) $$z_t^a - az_{xx}^a + cz^a = F, \quad 0 < x < 1, \quad t > 0$$

$$z^a(0,t) = 0, \quad z^a(1,t) = 0, \quad t > 0$$

$$z^a(x,0) = G(x), \quad 0 < x < 1$$

where $F = f - cr - r_t$ and $G = g_7 - r(x,0)$.

The condition $Q(a) \equiv -au_x^a(0,t^*) = g_8$ then becomes

(3.4) $$Q(a) \equiv -az_x^a(0,t^*) + a[g_5(t^*) - g_6(t^*)] = g_8 .$$

By integrating (3.3), we get

$$az_x^a(x,t) - az_x^a(0,t) = -\int_0^x [F - cz^a - z_t^a](s)ds .$$

Integrating again and applying the boundary conditions, we have

$$az_x^a(0,t) = \int_0^1\int_0^x [F - cz^a - z_t^a](s)ds\,dx .$$

Applying the Schwarz inequality and setting $t = t^*$, we easily obtain

(3.5) $$|az_x^a(0,t^*)| \le \frac{2}{3}\|[F - cz^a - z_t^a](\cdot,t^*)\|_0 .$$

It then follows that $\lim_{a\to 0} Q(a) = 0$ since $\lim_{a\to 0}\|[F - cz^a - z_t^a](\cdot,t^*)\|_0 = 0$ by lemma 1.

We will now show using a standard energy argument that $\|z^a(\cdot,t^*)\|_0$ and $\|z_t^a(\cdot,t^*)\|_0 \to 0$ as $a \to \infty$. This coupled with the fact (3.1) that $g_5(t^*) - g_6(t^*) > 0$

will prove that $\lim_{a\to\infty} Q(a) = +\infty$. The existence of a positive solution to $Q(a) = g_8$ for positive g_8 will then follow from the continuity of $Q(a)$.

Multiplying equation (3.3) by z^a and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|z^a(\cdot,t)\|_0^2 + a\|z_x(\cdot,t)\|_0^2 + \|(\sqrt{c}z)(\cdot,t)\|_0^2$$

$$= (F,z^a) \le \|F(\cdot,t)\|_0\|z^a(\cdot,t)\|_0$$

$$\le \frac{1}{6a}\|F(\cdot,t)\|_0^2 + \frac{3a}{2}\|z^a(\cdot,t)\|_0^2$$

(using the Schwarz and arithmetic-geometric mean inequalities). Since $z^a(0,t) = 0$, we easily get $\|z_x^a(\cdot,t)\|_0^2 \ge 2\|z^a(\cdot,t)\|_0^2$ and so

$$\frac{1}{2}\frac{d}{dt}\|z^a(\cdot,t)\|_0^2 + \frac{a}{2}\|z^a(\cdot,t)\|_0^2 \le \frac{1}{6a}\|F(\cdot,t)\|_0^2 .$$

Hence

$$\frac{d}{dt}\left[e^{at}\|z^a(\cdot,t)\|_0^2\right] \le \frac{1}{3a} e^{at}\|F(\cdot,t)\|_0^2$$

and after integrating we get

$$\|z^a(\cdot,t)\|_0^2 \le e^{-at}\|z^a(\cdot,0)\|_0^2 + \frac{1}{3a}\int_0^t e^{a(s-t)}\|F(\cdot,s)\|_0^2\,ds .$$

Since $t^* > 0$, we obtain $\lim_{a\to\infty}\|z^a(\cdot,t^*)\|_0 = 0$.

To obtain a similar result for z_t^a, we differentiate (3.3) to obtain the following initial boundary value problem for z_t^a (recall that $G(0) = G(1) = 0$ by (H0)).

(3.6)
$$[z_t^a]_t - a[z_t^a]_{xx} + cz_t^a = -c_t z^a + F_t$$

$$z_t^a(0,t) = 0, \quad z_t^a(1,t) = 0$$

$$z_t^a(x,0) = aG'' - c(x,0)G + F(x,0)$$

Using the same argument as in the previous case, we obtain

$$\|z_t^a(\cdot,t)\|_0^2 \le e^{-at}\|z_t^a(\cdot,0)\|_0^2 + \frac{1}{3a}\int_0^t e^{a(s-t)}\|[-c_t z^a + F_t](\cdot,s)\|_0^2\,ds .$$

It easily follows that $\lim_{a\to\infty}\|z_t^a(\cdot,t^*)\|_0 = 0$.

To obtain a continuous dependence result it is convenient to rewrite $u^a = y^a + w^a + g_7$, where y satisfies

$$(3.7) \qquad y_t^a - a y_{xx}^a = 0, \quad 0 < x < 1, \quad t > 0$$

$$y^a(0,t) = g_5(t) - g_5(0), \quad y^a(1,t) = 0, \quad t > 0$$

and
$$y^a(x,0) = 0, \quad 0 \le x \le 1,$$

and w^a satisfies

$$(3.8) \qquad w_t^a - a w_{xx}^a + c w^a = f - c y^a + a g_7'' - c g_7, \quad 0 < x < 1, \quad t > 0$$

$$w^a(0,t) = 0, \quad w^a(1,t) = g_6(t) - g_6(0), \quad t \ge 0$$

$$w^a(x,0) = 0, \quad 0 \le x \le 1 .$$

Then

$$Q(a) \equiv -a u_x^a(0,t^*) = -a y_x^a(0,t^*) - a w_x^a(0,t^*) - a g_7'(0)$$

and so

$$Q'(a) = \frac{\partial}{\partial a}[-a y_x^a(0,t^*)] + \frac{\partial}{\partial a}[-a w_x^a(0,t^*)] - g_7'(0) .$$

Now it is not difficult to show that

$$\frac{\partial}{\partial a}\left[-a w_x^a(0,t^*)\right] = -a \frac{\partial}{\partial a} w_x^a(0,t^*) - w_x^a(0,t^*) = -\frac{\partial}{\partial x}\left[a \frac{\partial w^a}{\partial a} + w^a\right]\Big|_{(0,t^*)}$$

where $\frac{\partial w^a}{\partial a}$ satisfies the partial differential equation

$$(3.9) \qquad \frac{\partial}{\partial t}\left[\frac{\partial w^a}{\partial a}\right] - a \frac{\partial^2}{\partial x^2}\left[\frac{\partial w^a}{\partial a}\right] + c\left[\frac{\partial w^a}{\partial a}\right] = w_{xx}^a - c \frac{\partial y^a}{\partial a} + g_7''$$

subject to homogeneous initial and boundary data, and $\frac{\partial y^a}{\partial a}$ satisfies the partial differential equation

$$(3.10) \qquad \frac{\partial}{\partial t}\left[\frac{\partial y^a}{\partial a}\right] - a \frac{\partial^2}{\partial x^2}\left[\frac{\partial y^a}{\partial a}\right] = y_{xx}^a = \frac{1}{a} y_t^a$$

also subject to homogeneous initial and boundary data.

Setting $p^a = a \frac{\partial w^a}{\partial a} + w^a$, we have that

$$Q'(a) = -\frac{\partial}{\partial x} p^a(0,t^*) + \frac{\partial}{\partial a}[-a y_x^a(0,t^*)] - g_7'(0)$$

where p^a satisfies

$$p_t^a - ap_{xx}^a + cp^a = aw_{xx}^a - ac\frac{\partial y^a}{\partial a} + ag_7'' + w_t^a - aw_{xx}^a + cw^a \tag{3.11}$$

$$= w_t^a + cw^a + ag_7'' - ac\frac{\partial y^a}{\partial a}$$

with $p^a(0,t) = 0$, $p^a(1,t) = g_6(t) - g_6(0)$, and $p^a(x,0) = 0$.

We now show that $p(x,t) \le 0$ for all $0 \le x \le 1$, $t \ge 0$. Since $p^a(0,t) = 0$, this will imply that $-\frac{\partial}{\partial x} p^a(0,t^*) \ge 0$. To do so we observe that by hypotheses (H3) and (H4) $g_6(t) - g_6(0) \le 0$ and $ag_7'' \le 0$. Hence the desired result will follow immediately from the maximum principle (lemma 3) if we can show that $w_t^a \le 0$, $w^a \le 0$, and $\frac{\partial y^a}{\partial a} \ge 0$, $(x,t) \in (0,1) \times (0,T]$.

Now using hypotheses (H1)-(H5) it easily follows from the maximum principle that $y^a \ge 0$ and hence that $w^a \le 0$ for all $(x,t) \in I_T$. To show that $w_t^a \le 0$ and $\frac{\partial y}{\partial a} \ge 0$ we observe that w_t^a satisfies the initial boundary value problem

$$\frac{\partial}{\partial t}[w_t^a] - a\frac{\partial^2}{\partial x^2}[w_t^a] + c[w_t^a] = -c_t w^a + f_t - cy_t^a - c_t y^a - c_t g_7 \tag{3.12}$$

$$w_t^a(0,t) = 0,\quad w_t^a(1,t) = g_6'(t)\ ,\quad \text{and}$$

$$w_t^a(x,0) = f(x,0) - c(x,0)g_7(x) + ag_7''(x)$$

and y_t^a satisfies:

$$\frac{\partial}{\partial t}[y_t^a] - a\frac{\partial^2}{\partial x^2}[y_t^a] = 0 \tag{3.13}$$

$$y_t^a(0,t) = g_5'(t),\quad y_t^a(1,t) = 0,\quad \text{and}\quad y_t^a(x,0) = 0\ .$$

By applying the maximum principle one easily shows that for all $(x,t) \in (0,1) \times (0,T]$

(3.14) $y_t^a \ge 0$ (using (H2))

(3.15) $y^a \le (1-x)[g_5(t) - g_5(0)]$ (using (H2))

(3.16) $w_t^a \le 0$ (using (H1), (H3), (H4), (H5), (H6), (H7), and (3.15))

and

(3.17) $\frac{\partial y^a}{\partial a} \ge 0$ (using (3.14)).

Thus we have

$$Q'(a) \geq \frac{\partial}{\partial a}[-ay_x(0,t^*)] - g_7'(0) .$$

Now from [2] we have that

$$(3.18) \qquad y_x^a(0,t^*) = -\int_0^{t^*} M(0,a(t^* - \tau)g_5'(\tau)d\tau$$

$$(3.19) \qquad \text{where } M(x,\sigma) = \pi^{-1/2}\sigma^{-1/2}\sum_{n=-\infty}^{\infty} \exp\left[-\frac{(x-2n)^2}{4\sigma}\right], \quad \sigma > 0 .$$

Hence

$$\frac{\partial}{\partial a}[-ay_x^a(0,t^*)] = \int_0^{t^*} M(0,a(t^* - \tau))g_5'(\tau)d\tau + \int_0^{t^*} M_\sigma(0,a(t^* - \tau))a(t - \tau)g_5'(\tau)d\tau .$$

But

$$M(x,\sigma) + \sigma M_\sigma(x,\sigma) = \frac{1}{2}\pi^{-1/2}\sigma^{-1/2}\sum_{n=-\infty}^{\infty} \exp\left[-\frac{(x-2n)^2}{4\sigma}\right]$$

$$+ \pi^{-1/2}\sigma^{-1/2}\sum_{n=-\infty}^{\infty} \exp\left[-\frac{(x-2n)^2}{4\sigma}\right] \cdot \frac{(x-2n)^2}{4\sigma}$$

and so

$$M(0,\sigma) + \sigma M_\sigma(0,\sigma) \geq \frac{1}{2}\pi^{-1/2}\sigma^{-1/2} .$$

Thus

$$\frac{\partial}{\partial a}[-ay_x^a(0,t^*)] \geq \frac{1}{2}\int_0^{t^*} \frac{g_5'(\tau)}{\sqrt{\pi a(t^* - \tau)}} d\tau$$

and so for all $0 < a \leq a^*$

$$(3.20) \qquad Q'(a) \geq \frac{1}{2}\int_0^{t^*} \frac{g_5'(\tau)}{\sqrt{\pi a^*(t^* - \tau)}} d\tau - g_7'(0)$$

$\equiv K(a^*) > 0$ (by (H8) and (H9)). (3.2) follows immediately from the mean value theorem.

We remark that when $f \equiv 0$, $c \equiv 0$, and $g_7 \equiv 0$, we are exactly in the situation analyzed in [2], where the result of Theorem 1 for the above special case was first obtained.

In the derivation of the error estimate we shall require a priori upper and lower bounds for the solution a. We now prove:

Lemma 4: Assume the conditions of Theorem 1 are satisfied. Then $0 < a_* \leq a \leq a^*$ where

$$a^* = \max\left\{1, \left[g_8/\left(\int_0^{t^*} \frac{g_5'(\tau)}{\sqrt{\pi(t^* - \tau)}}\, d\tau - g_7'(0)\right)\right]^2\right\}$$

and

$$a_* = \min\left\{\frac{1}{4}, \left(g_8/\left[\frac{2}{3}K_6^{t^*} + g_5(t^*) - g_6(t^*)\right]\right)^4\right\}$$

(and $K_6^{t^*}$ is a constant defined in lemma 1).

Proof: In the proof of Theorem 1 we showed that

$$g_8 = Q(a) \equiv -ay_x^a(0,t^*) - aw_x^a(0,t^*) - ag_7'(0)$$

and that $w^a(x,t) \leq 0$, $(x,t) \in I_T$. Since $w^a(0,t) = 0$ it follows that $-aw_x^a(0,t^*) \geq 0$ and so $g_8 \geq -ay_x^a(0,t^*) - ag_7'(0)$. Now from (3.18) and (3.19) we have

$$-ay_x^a(0,t^*) \geq a \int_0^{t^*} \frac{g_5(\tau)}{\sqrt{\pi a(t^* - \tau)}}\, d\tau$$

and so

$$g_8 \geq \sqrt{a} \int_0^{t^*} \frac{g_5'(\tau)}{\sqrt{\pi(t^* - \tau)}}\, d\tau - ag_7'(0)$$

If $a \geq 1$ then

$$g_8 \geq \sqrt{a}\left\{\int_0^{t^*} \frac{g_5(\tau)}{\sqrt{\pi(t^* - \tau)}}\, d\tau - g_7'(0)\right\}$$

and so

$$a \leq \max\left\{1, \left[g_8/\left(\int_0^{t^*} \frac{g_5(\tau)}{\sqrt{\pi(t^* - \tau)}}\, d\tau - g_7'(0)\right)\right]^2\right\} \equiv a^* .$$

To get a lower bound we observe that from (3.4) and (3.5) we have

$$g_8 \leq \frac{2}{3}\|[F - cz^a - z_t^a](\cdot,t^*)\|_0 + a[g_5(t^*) - g_6(t^*)] \leq \frac{2}{3} K_6^{t^*} a^{1/4} + a[g_5(t^*) - g_6(t^*)]$$

for $0 < a < 1/4$ (by lemma 1). Hence for $0 < a < 1/4$

$$g_8 \leq \left[\frac{2}{3} K_6^{t^*} + g_5(t^*) - g_6(t^*)\right]a^{1/4}$$

and so

$$a \geq \min\left\{\frac{1}{4}, \left(g_8/\left[\frac{2}{3} K_6^{t^*} + g_5(t^*) - g_6(t^*)\right]\right)^4\right\} \equiv a_* .$$

4. DESCRIPTION OF THE APPROXIMATE PROBLEM AND ERROR ESTIMATES FOR PROBLEM (P).

It is convenient to descirbe the approximate problem in somewhat abstract terms since part of the numerical procedure involves the approximation of the forward problem (1.5)-(1.7) for which many numerical schemes are available.

For a given let U^a denote an approximate solution of (1.5)-(1.7) corresponding to meshes of width h and Δt in the x and t directions respectively and satisfying an error estimate of the form

$$|u_x^\beta(0,t^*) - U_x^\beta(0,t^*)| \leq C(b_*,b^*)\varphi(h,\Delta t) \tag{4.1}$$

for all $\beta \in [b_*,b^*]$ where $\varphi(h,\Delta t)$ satisfies $\lim\limits_{\substack{h\to 0\\ \Delta t\to 0}} \varphi(h,\Delta t) = 0$ (for some schemes this type of estimate may only hold when some relation exists between the size of h and Δt, e.g. $\Delta t \leq Ch^2$).

The approximate problem is then to determine a constant α such that

$$\tilde{Q}(\alpha) \equiv -\alpha U_x^\alpha(0,t^*) = g_8 . \tag{4.2}$$

Since this is just a nonlinear equation in α, we can solve it by any standard iterative method where of course, in order to evaluate $\tilde{Q}(\alpha_n)$ we must find U^{α_n}, the approximate solution to the forward problem (1.5)-(1.7) (with a replaced by α_n).

We remark that for finite difference schemes the derivative $U_x^\beta(0,t^*)$ may be replaced by a difference quotient such as $[U^\beta(h,t^*) - U^\beta(0,t^*)]/h$, and Theorem 2 (to follow) easily amended to account for this change.

For approximation schemes satisfying (4.1), we will first show the existence of a solution to the approximate problem and derive an abstract error estimate. We then give specific error estimates for two examples where an estimate of the type (4.1) can be easily derived from results appearing in the literature.

<u>Theorem 2</u>: Suppose the hypotheses of Theorem 1 are satisfied and that (a,u^a) is the solution of problem (1.5)-(1.8). For a given let U^a denote an approximate solution of (1.5)-(1.7) satisfying condition (4.1). Let $0 < b_* < a_* < a^* < b^*$ where a_*,a^* are the bounds on a derived in lemma 4. Then for h and Δt

sufficiently small there exists an $\alpha \in [b_*, b^*]$ such that the pair (α, U^α) satisfies (4.2) and for all such solutions we have

(4.3) $$|a - \alpha| \le C(b_*, b^*)\varphi(h, \Delta t) .$$

Proof: By definition,

$$\tilde{Q}(b_*) = -b_* U_x^{b_*}(0,t^*) = -b_* u_x^{b_*}(0,t^*) + b_*[u_x^{b_*}(0,t^*) - U_x^{b_*}(0,t^*)]$$

$$\le Q(b_*) + C(b_*, b^*)\varphi(h, t) \qquad \text{(using (4.1))} .$$

Now

$$Q(b_*) = Q(b_*) - Q(a) + q_8 \le q_8 - K(a^*)(a - b_*)$$

(using (3.20) and the fact that $a_* \le a$). Hence

$$\tilde{Q}(b_*) \le q_8 - [K(a^*)(a - b_*) - C(b_*, b^*)\varphi(h, \Delta t)] < q_8 \quad \text{for } h, \Delta t \text{ sufficiently small}$$

since $\varphi(h, \Delta t) \to 0$ as $h, \Delta t \to 0$. Similarly,

$$\tilde{Q}(b^*) \ge Q(b^*) - C(b_*, b^*)\varphi(h, \Delta t) \ge q_8 + K(b^*)(b^* - a) - C(b_*, b^*)\varphi(h, \Delta t)$$

$$> q_8 \quad \text{for } h, \Delta t \text{ sufficiently small.}$$

By the continuity of $\tilde{Q}(\alpha)$, there exists an $\alpha \in [b_*, b^*]$ such that $\tilde{Q}(\alpha) = q_8$ for all $h, \Delta t$ sufficiently small.

To derive the error estimate we choose $b = \alpha$ in (3.2) to obtain

$$|a - \alpha| \le \frac{1}{K(b^*)} |Q(a) - Q(\alpha)| \le \frac{1}{K(b^*)} |q_8 - Q(\alpha)| = \frac{1}{K(b^*)} |\tilde{Q}(\alpha) - Q(\alpha)|$$

$$= \frac{\alpha}{K(b^*)} |U_x^\alpha(0,t^*) - u_x^\alpha(0,t^*)| \le \frac{b^*}{K(b^*)} C(b_*, b^*)\varphi(h, \Delta t)$$

(using (4.1) and the fact that $\alpha \in [b_*, b^*]$).

As an application of Theorem 2 we shall consider the Crank-Nicholson-Galerkin approximation to (1.5)-(1.7), which may be described as follows: Let $h = 1/N$, where N is a positive integer and set

$$S_h^k = \{v \in C^0[0,1] : v \in P_{k-1}(I_j), j = 1, \ldots, N\} ,$$

where $I_j = ([j-1]h, jh)$ and $P_{k-1}(I)$ is the set of all polynomials of degree $\le k-1$ defined on I. Define $\overset{\circ}{S}{}_h^k = \{v \in S_h^k : v(0) = v(1) = 0\}$. Now let $\Delta t = T/J$ where J is a positive integer and set $t_n = n\Delta t$, $n = 0, 1, \ldots, J$.

For a function F defined at times t_n we denote by F_n the function $F(t_n)$. We shall also use

$$F_{n+1/2} = (F_{n+1} + F_n)/2$$

$$(\text{Note } F_{n+1/2} \neq F(t_{n+1/2})) \text{ and}$$

$$\partial_t F_{n+1/2} = (F_{n+1} - F_n)/\Delta t \ .$$

Our approximate problem is then described as follows:

Determine a constant α and a sequence $\left\{U_n^\alpha\right\}_{n=0}^{J}$ where $U_n^\alpha = Z_n^\alpha + r_n$, $Z_n^\alpha \in \overset{\circ}{S}{}_h^k$ satisfies

$$(4.4) \qquad (\partial_t Z^\alpha_{n+1/2}, V) + a(\frac{\partial}{\partial x} Z^\alpha_{n+1/2}, \frac{\partial}{\partial x} V) + (c(\cdot,[n+\frac{1}{2}]\Delta t) Z^\alpha_{n+1/2}, V) = (F_{n+1/2}, V)$$

for all $V \in \overset{\circ}{S}{}_h^k$, $n \geq 0$ and

$$(4.5) \qquad a(\frac{\partial}{\partial x}[Z_0^\alpha - G], \frac{\partial}{\partial x} V) + (c(\cdot,0)[Z_0^\alpha - G], V) = 0 \quad \text{for all } V \in \overset{\circ}{S}{}_h^k,$$

where

$$r = (1-x)g_5(t) + xg_6(t) \ ,$$

$$F = f - cr - r_t, \text{ and}$$

$$G = g_7(x) - r(x,0) \ ,$$

and $-\alpha U_x^\alpha(0,t^*) = g_8$. ($Z_n^\alpha$ is the Crank-Nicolson-Galerkin approximation to z^α).

For the case when $c = c(x)$, Wheeler [15] has proved that if z (defined by (3.3)) belongs to $L_\infty(W^{k,\infty})$, $\frac{\partial z}{\partial t}$ belongs to $L_2(H^k)$ and $\frac{\partial^3 z}{\partial t^3}$ belongs to $L_2(L_2)$, then there are constants C and $\tau_0 > 0$ such that for all $0 < \Delta t < \tau_0$

$$(4.6) \quad |z^\alpha(\cdot,t_n) - Z_n^\alpha|_{L_\infty} \leq C[h^k(\|z\|_{L_\infty(W^{k,\infty})} + \|\frac{\partial z}{\partial t}\|_{L_2(H^k)}) + (\Delta t)^2 \|\frac{\partial^3 z}{\partial t^3}\|_{L_2(L_2)}], \quad n \geq 0 \ .$$

Although the constant C will depend on α it is $\leq C(b_*,b^*)$ for $\alpha \in [b_*,b^*]$.

Remarks: In [14] Wahlbin showed that an estimate similar to (4.6) could be derived for more easily computable choices of Z_0 than (4.5) and also for subspaces with higher interelement continuity. We further note that it is possible it extend the results in [15] to the case $c = c(x,t)$ for the problem we are considering. (A proof of this fact is included in the appendix as lemma 6).

To get an estimate of the type (4.1) we use an inverse condition satisfied by elements $V \in S_h^k$, i.e.,

$$\|V_x\|_{L_\infty} \le \frac{C}{h}\|V\|_{L_\infty} . \tag{4.7}$$

Letting V_I be the interpolate of $z^\alpha(\cdot,t_n)$ in S_h^k, we get

$$\begin{aligned}
\left\|\frac{\partial}{\partial x}[u^\alpha(\cdot,t_n) - U_n^\alpha]\right\|_{L_\infty} &= \left\|\frac{\partial}{\partial x}[z^\alpha(\cdot,t_n) - Z_n^\alpha]\right\|_{L_\infty} \\
&\le \left\|\frac{\partial}{\partial x}[z^\alpha(\cdot,t_n) - V_I]\right\|_{L_\infty} + \frac{C}{h}\|V_I - Z_n^\alpha\|_{L_\infty} \\
&\le \left\|\frac{\partial}{\partial x}[z^\alpha(\cdot,t_n) - V_I]\right\|_{L_\infty} + \frac{C}{h}\|V_I - z^\alpha(\cdot,t_n)\|_{L_\infty} + \frac{C}{h}\|z^\alpha(\cdot,t_n) - Z_n^\alpha\|_{L_\infty} \\
&\le Ch^{k-1}\|z^\alpha(\cdot,t_n)\|_{W^{k,\infty}} + \frac{C}{h}\|z^\alpha(\cdot,t_n) - Z_n^\alpha\|_{L_\infty}
\end{aligned}$$

(using the standard approximation properties of subspace S_h^k).

Applying (4.6), we will then get the desired estimate with $\varphi(h,\Delta t) = h^{k-1} + (\Delta t)^2/h$ provided that the solution z^α has the indicated regularity for <u>all</u> $\alpha \in [b_*,b^*]$. Using Theorem 5.3 in Lions [8], this will be the case if the following compatibility conditions are satisfied.

$$G(x) = 0, \quad x = 0,1 , \tag{4.8}$$

$$\alpha G''(x) + c(x)G(x) + F(x,0) = 0, \quad x = 0,1 , \tag{4.9}$$

and

$$\begin{aligned}
&\alpha[\alpha G''(x) + c(x)G(x) + F(x,0)]'' \\
&+ c(x)[\alpha G''(x) + c(x)G(x) + F(x,0)] \\
&+ F_t(x,0) = 0, \quad x = 0,1 .
\end{aligned} \tag{4.10}$$

Since we require that conditions (4.9) and (4.10) hold for all $\alpha \in [b_*,b^*]$ these two conditions actually lead to the more restrictive conditions that

$$G''(x) = 0, \quad x = 0,1 , \tag{4.11}$$

$$c(x)G(x) + F(x,0) = 0, \quad x = 0,1 \tag{4.12}$$

$$G^{(iv)}(x) = 0, \quad x = 0,1 \tag{4.13}$$

$$[c(x)G(x) + F(x,0)]'' = 0, \quad x = 0,1 \tag{4.14}$$

and

(4.15)
$$F_t(x,0) = 0, \quad x = 0,1 .$$

(For the case $c = c(x,t)$ the compatibility conditions become more complicated. See Theorem 6.2 of [8]).

Under these conditions, we get the error estimate

$$|a - \alpha| \le C(b_*, b^*)[h^{k-1} + (\Delta t)^2/h] \qquad (k \le 4) .$$

By making use of the results of [1] and observing that condition (4.1) only involves an estimate at time $t = t^* > 0$, the compatibility conditions stated previously can be weakened for a special case of the problem we are considering. Suppose that g_5 and g_6 are constants, $c = c(x)$, and $f = f(x)$. Let $\theta^\alpha(x)$ be the solution of the steady state problem

(4.16)
$$-\alpha \frac{d^2}{dx^2}\theta^\alpha + c\theta^\alpha = f, \quad 0 < x < 1$$

$$\theta^\alpha(0) = g_5, \quad \theta^\alpha(1) = g_6 .$$

Then $v^\alpha = u^\alpha - \theta^\alpha$ satisfies:

(4.17)
$$v_t^\alpha - \alpha v_{xx}^\alpha + cv^\alpha = 0, \quad 0 < x < 1, \quad t > 0$$

$$v^\alpha(0,t) = 0, \quad v^\alpha(1,t) = 0, \quad t > 0$$

$$v^\alpha(x,0) = g_7(x) - \theta^\alpha(x)$$

Now denote by $\Theta^\alpha \in S_h^k$ the standard Galerkin approximation to the steady state solution θ^α and by $\{V_n^\alpha\}$ the Crank-Nicolson-Galerkin approximation to v^α. It is shown in [1] that

(4.18)
$$|v^\alpha(\cdot,t_n) - V_n^\alpha|_{L_\infty} \le C(t_n)[h^k + (\Delta t)^2]\,||g_7 - \theta^\alpha||_0$$

provided $\Delta t \le Ch^2$. Since we have from [16] that

(4.19)
$$|\theta^\alpha - \Theta^\alpha|_{L_\infty} \le Ch^k ||\theta^\alpha||_{W^{k,\infty}} ,$$

it follows that for $U_n^\alpha = V_n^\alpha + \Theta^\alpha$

$$|u^{\alpha}(\cdot,t^*) - U^{\alpha}(\cdot,t^*)|_{L_\infty} \le C(b_*,b^*)[h^k + (\Delta t)^2]$$

and using the inverse condition as before that (4.1) holds with $\varphi(h,\Delta t) = h^{k-1} + (\Delta t)^2/h$ without requiring the corresponding compatibility conditions:

(4.20) $$[g_7''(x) - \theta''(x)] = 0, \quad x = 0,1$$

and

(4.21) $$c'(x)[g_7'(x) - \theta'(x)] = 0, \quad x = 0,1 .$$

Choosing S_h^k = the space of continuous piecewise cubics (i.e. $k = 4$) and $\Delta t = Ch^2$, we would then get

(4.22) $$|a - \alpha| \le C(b_*,b^*)h^3 .$$

We note that results analogous to (4.18) are obtained in [1] for a wide class of single step Galerkin approximations, and that error estimates corresponding to use of these schemes are obtained in a similar fashion.

5. WELL POSEDNESS OF PROBLEM (E).

We now turn to the problem of determining the unknown coefficient in the elliptic boundary value (1.9)-(1.10) subject to condition (1.11). In a manner similar to section 3, we shall first determine some simple conditions under which Problem (E) is well posed, i.e. there exists a unique positive solution to the equation $Q(a) \equiv a \frac{\partial u^a}{\partial n}(x^*) = g^*$ which is continuously dependent on g^* (we shall assume that c, f and g are fixed).

To do so we will make use of the function ω satisfying:

$$-\Delta\omega = 0 \quad \text{in } \Omega$$

$$\omega = g \quad \text{on } \Gamma .$$

We shall then assume that $c, f,$ and g are sufficiently smooth and that

(A1) $f - c\omega \leq 0, \quad x \in \Omega ,$

(A2) $\frac{\partial\omega}{\partial n}(x^*) > 0,$ and

(A3) $0 < c_1 \leq c(x) \leq c_2 .$

We note that by a form of the maximum principle (e.g. see [12], Theorem 8) if $\max_{x\in\Gamma} g(x) = g(x^*) > 0,$ and $g(x)$ is not identically constant, then $\frac{\partial\omega}{\partial n}(x^*) > 0.$ Hence this gives a simple sufficient condition for (A2) to hold.

We now prove the following:

Theorem 3: Suppose that hypotheses (A1)-(A3) are satisfied. Then for any $g^* > 0$ there exists a unique positive solution of $Q(a) \equiv a \frac{\partial u}{\partial n}(x^*) = g^*$. Furthermore, for all $a, b > 0$,

$$|a - b| \leq |Q(a) - Q(b)| / \frac{\partial\omega}{\partial n}(x^*) . \tag{5.1}$$

Proof. Write $u^a = z^a + \omega$. Then z^a satisfies

$$-a\Delta z^a + cz^a = f - c\omega \quad \text{in } \Omega \tag{5.2}$$

$$z^a = 0 \quad \text{on } \Gamma .$$

The condition $Q(a) \equiv a \frac{\partial u^a}{\partial n}(x^*) = g^*$ then becomes

$$Q(a) \equiv a \frac{\partial z^a}{\partial n}(x^*) + a \frac{\partial\omega}{\partial n}(x^*) = g^* . \tag{5.3}$$

Now from (5.2) we have $-a\Delta z^a = f - c\omega - cz^a$ and so

$$a\|z^a\|_{W^{2,4}} = \|-\Delta^{-1}[f - c\omega - cz^a]\|_{W^{2,4}} \leq \|-\Delta^{-1}\|_{\underline{\mathcal{L}}(L_4, W^{2,4}\cap H_0^1)} \|f - c\omega - cz^a\|_{L_4} .$$

By the Sobolev lemma $\left|\frac{\partial z^a}{\partial n}(x^*)\right| \leq C\|z^a\|_{W^{2,4}}$ and so

$$\left|a \frac{\partial z^a}{\partial n}(x^*)\right| \leq C\|f - c\omega - cz^a\|_{L_4} . \tag{5.4}$$

It then follows that $\lim_{a\to 0} Q(a) = 0$ since $\|f - c\omega - cz^a\|_{L_4} \to 0$ as $a \to 0$ by lemma 2.

Now multiplying equation (5.2) by $[z^a]^3$ and integrating by parts, we have

$$3a\|z^a\nabla z^a\|_0^2 + \|\sqrt{c}\,[z^a]^2\|_0^2 = \left(f - c\omega, [z^a]^3\right)$$

$$\leq \|f - c\omega\|_{L_4} \|[z^a]^3\|_{L_{4/3}} \leq \|f - c\omega\|_{L_4} \|z^a\|_{L_4}^3 .$$

Hence $\|z^a\|_{L_4} \leq \frac{1}{c_1}\|f - c\omega\|_{L_4}$ (using A3). This implies that $\left|a \frac{\partial z^a}{\partial n}(x^*)\right|$ is bounded for all a and thus from (A2) and (5.3) it follows that $\lim_{a\to+\infty} Q(a) = +\infty$. The existence of a positive solution to $Q(a) = g^*$ for positive g^* follows from the continuity of $Q(a)$.

To prove continuous dependence, we differentiate $Q(a)$ to obtain

$$Q'(a) = \frac{\partial}{\partial a}\left[a \frac{\partial z^a}{\partial n}(x^*)\right] + \frac{\partial \omega}{\partial n}(x^*) .$$

Now it is not difficult to show that

$$\frac{\partial}{\partial a}\left[a \frac{\partial z^a}{\partial n}(x^*)\right] = a \frac{\partial}{\partial a}\frac{\partial z^a}{\partial n}(x^*) + \frac{\partial z^a}{\partial n}(x^*) = \frac{\partial}{\partial n}\left[a \frac{\partial z^a}{\partial a} + z^a\right](x^*)$$

where $\frac{\partial z^a}{\partial a}$ satisfies the boundary value problem

$$-a\Delta\left[\frac{\partial z^a}{\partial a}\right] + c\left[\frac{\partial z^a}{\partial a}\right] = \Delta z^a \quad \text{in } \Omega$$

$$\frac{\partial z^a}{\partial a} = 0 \quad \text{on } \Gamma .$$

Setting $p^a = a\frac{\partial z^a}{\partial a} + z^a$, we have that $Q'(a) = \frac{\partial p^a}{\partial n}(x^*) + \frac{\partial \omega}{\partial n}(x^*)$ where p^a satisfies

$$-a\Delta p^a + cp^a = a\Delta z^a - a\Delta z^a + cz^a \quad \text{in } \Omega \tag{5.5}$$

$$p^a = 0 \quad \text{on } \Gamma$$

Applying the maximum principle it follows from (A1) and (5.2) that $z^a \leq 0$ and hence from (5.5) that $p^a \leq 0$ in Ω. Since $P^a = 0$ on Γ, $\frac{\partial p^a}{\partial n}(x^*) \geq 0$ and so $Q'(a) \geq \frac{\partial \omega}{\partial n}(x^*)$. (5.1) now follows directly from the mean value theorem.

As in the parabolic problem we shall require a priori upper and lower bounds for the solution a in order to derive the error estimate. We now prove:

Lemma 5: Assume the conditions of Theorem 3 are satisfied. Then $0 < a_* \leq a \leq a^*$ for some constants a_*, a^* depending only on the data of Problem (E).

Proof: In the proof of Theorem 3 we showed that

$$g^* = Q(a) \equiv a \frac{\partial z^a}{\partial n}(x^*) + a \frac{\partial \omega}{\partial n}(x^*)$$

and that $z^a \leq 0$ in Ω. Since $z^a = 0$ on Γ, it follows that $\frac{\partial z^a}{\partial n}(x^*) \geq 0$ and so $g^* \geq a \frac{\partial \omega}{\partial n}(x^*)$. Hence by (A2) $a \leq g^*/\frac{\partial \omega}{\partial n}(x^*) \equiv a^*$.

To get a lower bound we observe that from (5.3) and (5.4)

$$g^* \leq C\|f - c\omega - cz^a\|_{L_4} + a \frac{\partial \omega}{\partial n}(x^*) \leq ca^{1/8} + a \frac{\partial \omega}{\partial n}(x^*)$$

for $0 < a < \varepsilon_0^2$ (by lemma 2). Hence for $0 < a < \varepsilon_0^2$ (ε_0^2 defined in lemma 2) $g^* \leq Ca^{1/8}$ and so

$$a \geq \min\{\varepsilon_0^2, [g^*/C]^8\} \equiv a_* .$$

Remarks: We now make two comments about the requirement in (A3) that $c(x) \geq c_1 > 0$. As evident from the proof of Theorem 3, the reason for this restriction was to prove existence of a solution for all positive g^*. We note however that if c only satisfies $c(x) \geq 0$, then it is still true that $Q'(b) > 0$ for $b > 0$. Hence if we also know that $Q(a_1) = g_1^*$ and $Q(a_2) = g_2^*$ with $0 < a_1 < a_2$, then for any $g_1^* \leq g^* \leq g_2^*$, there will exist an $a_1 \leq a \leq a_2$ such that $Q(a) = g^*$. Since we would then have both the existence of a solution and a priori bounds on a without the use of lemma 2, the requirement that $c(x) \geq c_1 > 0$ could be weakened.

The second remark has to do with the case $c(x) \equiv 0$. In this case $z^a = z^1/a$ where z^1 satisfies

$$-\Delta z^1 = f \quad \text{in } \Omega$$
$$z^1 = 0 \quad \text{on } \Gamma .$$

Condition (5.3) becomes

$$g^* = \frac{\partial z^1}{\partial n} (x^*) + a \frac{\partial \omega}{\partial n} (x^*)$$

and so

$$a = [g^* - \frac{\partial z^1}{\partial n} (x^*)] / \frac{\partial \omega}{\partial n} (x^*) .$$

Thus the problem is trivial and in fact we see that if $\frac{\partial \omega}{\partial n} (x^*) = 0$ then the problem will have no solution unless $g^* = \frac{\partial z^1}{\partial n} (x^*)$. In that case $u^a = z^1/a + \omega$ for any value of a and we have nonuniqueness.

6. DESCRIPTION OF THE APPROXIMATE PROBLEM AND ERROR ESTIMATES FOR PROBLEM (E).

As in Section 4, it is convenient to describe the approximate problem in abstract terms so that we may easily take into account the many numerical schemes available for solving the forward problem (1.9)-(1.10).

For a given, let U^a denote an approximate solution to (1.9)-(1.10) satisfying an error estimate of the form

$$(6.1) \qquad \left|\frac{\partial u^\beta}{\partial n}(x^*) - \frac{\partial U^\beta}{\partial n}(x^*)\right| \leq C(b_*,b^*)\varphi(h)$$

for all $\beta \in [b_*,b^*]$ where $\varphi(h)$ satisfies $\lim_{h\to 0} \varphi(h) = 0$. (In the applications, h might denote the size of a finite element mesh).

The approximate problem is then to determine a constant α such that

$$(6.2) \qquad \tilde{Q}(\alpha) \equiv \alpha \frac{\partial U^\alpha}{\partial n}(x^*) = g^* .$$

Once again this is just a nonlinear equation in α, easily solvable by standard iterative techniques.

Using essentially the same proof as in Theorem 2, we get:

Theorem 4: Suppose the hypotheses of Theorem 3 are satisfied and that (a,u^a) is the solution of Problem (E). For a given, let U^a denote an approximate solution of (1.9)-(1.10) satisfying condition (6.1). Let $0 < b_* < a_* < a^* < b^*$ where a_*,a^* are the bounds on a given in lemma 5. Then for h sufficiently small there exists an $\alpha \in [b_*,b^*]$ such that the pair (α,U^α) satisfies (6.2) and for all such solutions we have

$$(6.3) \qquad |a - \alpha| \leq C(b_*,b^*)\varphi(h) .$$

In order to apply Theorem 4, we need only choose the approximation scheme and determine the function $\varphi(h)$ of (6.1) Estimates of this type for various finite element methods follow easily from the work of Nitsche [9], Goldstein and Scott [6], Wahlbin [13] and others described in the references of those papers, provided we have the inverse condition $\|\nabla V\|_{L_\infty} \leq \frac{C}{h}\|V\|_{L_\infty}$ for all $V \in$ finite element subspace being used.

For the standard finite element method applied to (1.9)-(1.10), for example, with subspaces of piecewise polynomials of degree $k - 1$ we get that (6.1) holds with

$$\varphi(h) = \begin{cases} h^{k-1}, & k \geq 3 \\ h|\ell n h|, & k = 2 . \end{cases}$$

Hence using piecewise cubics for example, we would get $|a - \alpha| \leq Ch^3$.

7. APPENDIX.

Proof of lemma 1: For any $\frac{1}{2} > \varepsilon > 0$ define

$$\chi^{\varepsilon}(x) = \begin{cases} x/\varepsilon \quad , & 0 \leq x \leq \varepsilon \\ 1 \quad , & \varepsilon \leq x \leq 1 - \varepsilon \\ (1 - x)/\varepsilon, & 1 - \varepsilon \leq x \leq 1 . \end{cases}$$

Then one easily verifies that

$$0 \leq \chi^{\varepsilon} \leq 1 \tag{7.1}$$

$$\|1 - \chi^{\varepsilon}\|_0 \leq \sqrt{\frac{2\varepsilon}{3}} \quad \text{and} \tag{7.2}$$

$$\|\frac{\partial}{\partial x} \chi^{\varepsilon}\|_0 \leq 2/\sqrt{\varepsilon} . \tag{7.3}$$

Write $z^a = z_1^a + z_2^a$, where z_1^a satisfies

$$\frac{\partial}{\partial t} z_1^a - a \frac{\partial^2}{\partial x^2} z_1^a + cz_1^a = \chi^{\varepsilon}[z_t^0 + cz^0] \tag{7.4}$$

$$z_1^a(0,t) = 0, \quad z_1^a(1,t) = 0, \quad z_1^a(x,0) = G(x)$$

and z_2^a satisfies

$$\frac{\partial}{\partial t} z_2^a - a \frac{\partial^2}{\partial x^2} z_2^a + cz_2^a = F[1 - \chi^{\varepsilon}] \tag{7.5}$$

$$z_2^a(0,t) = 0, \quad z_2^a(1,t) = 0, \quad z_2^a(x,0) = 0 .$$

From (7.5) it follows that

$$(\frac{\partial}{\partial t} z_2^a, z_2^a) + a(\frac{\partial}{\partial x} z_2^a, \frac{\partial}{\partial x} z_2^a) + (cz_2^a, z_2^a)$$

$$= (F[1 - \chi^{\varepsilon}], z_2^a) \leq \sqrt{\frac{2\varepsilon}{3}} \|F(\cdot,t)\|_{L_\infty} \|z_2^a(\cdot,t)\|_0$$

(using (7.2))

$$\leq \frac{1}{2}\left[\sqrt{\frac{2\varepsilon}{3}} \|F(\cdot,t)\|_{L_\infty}\right]^2 + \frac{1}{2} \|z_2^a(\cdot,t)\|_{L_\infty}^2 .$$

Hence

$$\frac{\partial}{\partial t} \|z_2^a(\cdot,t)\|_0^2 \leq \left[\sqrt{\frac{2\varepsilon}{3}} \|F(\cdot,t)\|_{L_\infty}\right]^2 + \|z_2^a(\cdot,t)\|_0^2 .$$

Applying Gronwall's inequality, we get

$$\|z_2^a(\cdot,t)\|_0 \le \sqrt{\frac{2\varepsilon}{3}}\,\{\int_0^t e^{t-s}\ \|F(\cdot,s)\|_{L_\infty}^2\,ds\}^{1/2} .$$

Now using (7.4), it follows that

$$(\frac{\partial}{\partial t}[z_1^a - \chi^\varepsilon z^0],\ z_1^a - \chi^\varepsilon z^0) + a(\frac{\partial}{\partial x} z_1^a,\ \frac{\partial}{\partial x} z_1^a) + (c[z_1^a - \chi^\varepsilon z^0],\ z_1^a - \chi^\varepsilon z^0)$$

$$= a(\frac{\partial}{\partial x} z_1^a,\ \frac{\partial}{\partial x}[\chi^\varepsilon z^0]) \le a\|\frac{\partial}{\partial x} z_1^a(\cdot,t)\|_0^2 + \frac{a}{4}\ \|\frac{\partial}{\partial x}[\chi^\varepsilon z^0(\cdot,t)]\|_0^2 .$$

Hence

$$\frac{\partial}{\partial t}\ \|[z_1^a - \chi^\varepsilon z^0](\cdot,t)\|_0^2 \le \frac{a}{2}\ \|\frac{\partial}{\partial x}[\chi^\varepsilon z^0(\cdot,t)]\|_0^2$$

and so

$$\|[z_1^a - \chi^\varepsilon z^0](\cdot,t)\|_0^2 \le \|[1-\chi^\varepsilon]G(x)\|_0^2 + \frac{a}{2}\int_0^t \|\frac{\partial}{\partial x}[\chi^\varepsilon z^0(\cdot,s)]\|_0^2\,ds .$$

Now

$$\|\frac{\partial}{\partial x}[\chi^\varepsilon z^0(\cdot,s)]\|_0 \le \|\chi^\varepsilon \frac{\partial}{\partial x} z^0(\cdot,s)\|_0 + \|z^0(\cdot,s)\ \frac{\partial}{\partial x}\chi^\varepsilon\|_0$$

$$\le \|\frac{\partial}{\partial x} z^0(\cdot,s)\|_0 + \frac{2}{\sqrt{\varepsilon}}\|z^0(\cdot,s)\|_{L_\infty}$$

(using (7.1) and (7.3)). Hence

$$\|[z_1^a - \chi^\varepsilon z^0](\cdot,t)\|_0^2 \le \left[\sqrt{\frac{2\varepsilon}{3}}\ \|G\|_{L_\infty}\right]^2 + a\int_0^t \|\frac{\partial}{\partial x} z^0(\cdot,s)\|_0^2\,ds$$

$$+ a\int_0^t \left[\frac{2}{\sqrt{\varepsilon}}\ \|z^0(\cdot,s)\|_{L_\infty}\right]^2 ds$$

and so

$$\|[z_1^a - \chi^\varepsilon z^0](\cdot,t)\|_0 \le \sqrt{\frac{2\varepsilon}{3}}\ \|G\|_{L_\infty} + \sqrt{a}\,\{\int_0^t \|\frac{\partial}{\partial x} z^0(\cdot,s)\|_0^2\,ds\}^{1/2}$$

$$+ 2\sqrt{\frac{a}{\varepsilon}}\,\{\int_0^t \|z^0(\cdot,s)\|_{L_\infty}^2\,ds\}^{1/2} .$$

Applying the triangle inequality, we get

$$\|[z^a - z^0](\cdot,t)\|_0 \le \|[z_1^a - \chi^\varepsilon z^0](\cdot,t)\|_0 + \|z_2^a(\cdot,t)\|_0$$

$$+ \|[\chi^\varepsilon - 1]z^0(\cdot,t)\|_0 \le K_1^t\sqrt{\varepsilon} + K_2^t\sqrt{a} + K_3^t\sqrt{\frac{a}{\varepsilon}}$$

where

$$K_1^t = \sqrt{\frac{2}{\varepsilon}}\,\|G\|_{L_\infty} + \sqrt{\frac{2}{3}}\,\{\int_0^t e^{t-s}\|F(\cdot,s)\|_{L_\infty}^2 ds\}^{1/2} + \sqrt{\frac{2}{\varepsilon}}\,\|z^0(\cdot,t)\|_{L_\infty} ,$$

$$K_2^t = \{\int_0^t \|\frac{\partial}{\partial x} z^0(\cdot,s)\|_0^2 ds\}^{1/2} ,$$

and

$$K_3^t = 2\{\int_0^t \|z^0(\cdot,s)\|_{L_\infty}^2 ds\}^{1/2} .$$

Choosing $\varepsilon = \sqrt{a}$, we have for $0 < a < 1/4$ that

$$\|[z^a - z^0](\cdot,t)\|_0 \leq K_4^t a^{1/4} \tag{7.6}$$

where

$$K_4^t = K_1^t + K_2^t/\sqrt{2} + K_3^t .$$

To obtain an estimate for $\|[z_t^a - z_t^0](\cdot,t)\|_0$ we observe that since $G \in C^2[0,1]$ with $G(0) = G(1) = 0$, z_t^a satisfies the initial boundary value problem

$$\frac{\partial}{\partial t} z_t^a - a\frac{\partial^2}{\partial x^2} z_t^a + cz_t^a = F_t - c_t z^a$$

$$z_t^a(0,t) = 0, \quad z_t^a(1,t) = 0$$

$$z_t^a(x,0) = aG''(x) + F(x,0) - c(x,0)G(x) ,$$

and z_t^0 satisfies

$$\frac{\partial}{\partial t} z_t^0 + cz_t^0 = F_t - c_t z^0$$

$$z_t^0(x,0) = F(x,0) - c(x,0)G(x) .$$

We now apply the previous argument to these problems, i.e. write

$$z_t^a = z_{t1}^a + z_{t2}^a \quad \text{where} \quad z_{t1}^a \quad \text{satisfies}$$

$$\frac{\partial}{\partial t} z_{t1}^a - a\frac{\partial^2}{\partial x^2} z_{t1}^a + cz_{t1}^a = \chi^\varepsilon[\frac{\partial}{\partial t} z_t^0 + cz_t^0]$$

$$z_{t1}^a(0,t) = 0, \quad z_{t1}^a(1,t) = 0 \tag{7.7}$$

$$z_{t1}^a(x,0) = aG''(x) + F(x,0) - c(x,0)G(x)$$

and z^a_{t2} satisfies

$$\frac{\partial}{\partial t} z^a_{t2} - a \frac{\partial^2}{\partial x^2} z^a_{t2} + cz^a_{t2} = [1 - \chi^\varepsilon][F_t - c_t z^0] - c_t[z^a - z^0] \tag{7.8}$$

$$z^a_{t2}(0,t) = 0, \quad z^a_{t2}(1,t) = 0, \quad z^a_{t2}(x,0) = 0 .$$

Proceeding as before, we obtain from (7.8) that

$$\|z^a_{t2}(\cdot,t)\|_0 \leq \sqrt{2}\left\{\int_0^t e^{t-s}\left[\sqrt{\frac{2\varepsilon}{3}}\ \|[F_t - c_t z^0\right](\cdot,s)\|^2_{L_\infty} ds\right\}^{1/2}$$

$$+ \sqrt{2}\ \{\int_0^t e^{t-s}\|c_t(\cdot,s)\|^2_{L_\infty}\|[z^a - z^0](\cdot,s)\|^2_0 ds\}^{1/2} .$$

Again following our previous argument, we get using (7.7) that

$$\|[z^a_{t1} - \chi^\varepsilon z^0_t](\cdot,t)\|_0 \leq a\|G''\|_0 + \sqrt{\frac{2\varepsilon}{3}}\ \|F(\cdot,0) - c(\cdot,0)G\|_{L_\infty}$$

$$+ \sqrt{a}\ \{\int_0^t \|\frac{\partial}{\partial x} z^0_t(\cdot,s)\|^2_0 ds\}^{1/2} + 2\sqrt{\frac{a}{\varepsilon}}\ \{\int_0^t \|z^0_t(\cdot,s)\|^2_{L_\infty} ds\}^{1/2} .$$

Applying the triangle inequality,

$$\|[z^a_t - z^0_t](\cdot,t)\|_0 \leq \|[z^a_{t1} - \chi^\varepsilon z^0_t](\cdot,t)\|_0 + \|z^a_{t2}(\cdot,t)\|_0 + \|[\chi^\varepsilon - 1]z^0_t(\cdot,t)\|_0 .$$

Choosing $\varepsilon = \sqrt{a}$, inserting inequality (7.6), and collecting terms we will have for some constant K^t_5 that for all $0 < a < 1/4$

$$\|[z^a_t - z^0_t](\cdot,t)\|_0 \leq K^t_5 a^{1/4} .$$

Inequality (2.3) now follows from the definition of z^0 and the triangle inequality.

Proof of lemma 2: For $\varepsilon > 0$ define

$$\chi^\varepsilon(x) = \begin{cases} 1, & \rho(x,\Gamma) \geq \varepsilon \\ \frac{1}{\varepsilon}\rho(x,\Gamma), & \rho(x,\Gamma) \leq \varepsilon \end{cases}$$

where $\rho(x,\Gamma)$ denotes the distance of a point $x \in \Omega$ to the boundary Γ. Since Ω is assumed sufficiently smooth there will exist an $1 > \varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $\chi^\varepsilon \in W^{1,4}$ and satisfies

(7.9) $$0 \le \chi^{\varepsilon} \le 1$$

(7.10) $$\|1 - \chi^{\varepsilon}\|_{L_4} \le C\varepsilon^{1/4} ,$$

and

(7.11) $$\|\nabla\chi^{\varepsilon}\|_{L_4} \le C\varepsilon^{-3/4} .$$

Now write $z^a = z_1^a + z_2^a$ where z_1^a satisfies

(7.12) $$(-a\Delta + c)z_1^a = (f - c\omega)\chi^{\varepsilon} \quad \text{in } \Omega$$

$$z_1^a = 0 \quad \text{on } \Gamma$$

and z_2^a satisfies

(7.13) $$(-a\Delta + c)z_2^a = (f - c\omega)(1 - \chi^{\varepsilon}) \quad \text{in } \Omega$$

$$z_2^a = 0 \quad \text{on } \Gamma .$$

From (7.13) it follows that

$$-a(\Delta z_2^a,[z_2^a]^3) + (cz_2^a,[z_2^a]^3) = ([f - c\omega][1 - \chi^{\varepsilon}],[z_2^a]^3)$$

and after integration by parts that

$$3a\|z_2^a\nabla z_2^a\|_0^2 + \|c^{1/4}z_2^a\|_{L_4}^4 \le \|f - c\omega\|_{L_\infty}\|1 - \chi^{\varepsilon}\|_{L_4}\|[z_2^a]^3\|_{L_{4/3}}$$

$$\le C\|f - c\omega\|_{L_\infty}\varepsilon^{1/4}\|z_2^a\|_{L_4}^3$$

(by 7.10)). Hence

(7.14) $$\|z_2^a\|_{L_4} \le C\|f - c\omega\|_{L_\infty}\varepsilon^{1/4}$$

(since $c(x) \ge c_1 > 0$).

Setting

$$\psi^{\varepsilon} = [\frac{f}{c} - \omega]\chi^{\varepsilon} ,$$

it follows from (7.12) that

$$-a(\Delta[z_1^a - \psi^{\varepsilon}],[z_1^a - \psi^{\varepsilon}]^3) + (c[z_1^a - \psi^{\varepsilon}],[z_1^a - \psi^{\varepsilon}]^3) = a(\Delta\psi^{\varepsilon},[z_1^a - \psi^{\varepsilon}]^3)$$

and after integration by parts that

$$3a\|[z_1^a - \psi^\varepsilon]\nabla[z_1^a - \psi^\varepsilon]\|_0^2 + \|c^{1/4}[z_1^a - \psi^\varepsilon]\|_{L_4}^4 = -3a(\nabla\psi^\varepsilon, [z_1^a - \psi^\varepsilon]^2\nabla[z_1^a - \psi^\varepsilon])$$

$$\leq 3a\|[z_1^a - \psi^\varepsilon]\nabla\psi^\varepsilon\|_0 \|[z_1^a - \psi^\varepsilon]\nabla[z_1^a - \psi^\varepsilon]\|_0$$

$$\leq \frac{3a}{4}\|[z_1^a - \psi^\varepsilon]\nabla\psi^\varepsilon\|_0^2 + 3a\|[z_1^a - \psi^\varepsilon]\nabla[z_1^a - \psi^\varepsilon]\|_0^2 .$$

Hence

$$\|z_1^a - \psi^\varepsilon\|_{L_4}^4 \leq \frac{3a}{4c_1}\|[z_1^a - \psi^\varepsilon]\nabla\psi^\varepsilon\|_0^2 \leq \frac{3a}{4c_1}\|z_1^a - \psi^\varepsilon\|_{L_4}^2\|\nabla\psi^\varepsilon\|_{L_4}^2$$

and so

(7.15)
$$\|z_1^a - \psi^\varepsilon\|_{L_4} \leq \sqrt{\frac{3a}{4c_1}}\,\|\nabla\psi^\varepsilon\|_{L_4} .$$

Now

(7.16)
$$\|\nabla\psi^\varepsilon\|_{L_4} \leq \|\chi^\varepsilon\nabla[\tfrac{f}{c} - \omega]\|_{L_4} + \|[\tfrac{f}{c} - \omega]\nabla\chi^\varepsilon\|_{L_4}$$

$$\leq \|\nabla[\tfrac{f}{c} - \omega]\|_{L_4} + C\|\tfrac{f}{c} - \omega\|_{L_\infty}\varepsilon^{-3/4}$$

(by (7.11)). Using the triangle inequality we get

$$\|f - c\omega - cz^a\|_{L_4} \leq c_2\{\|[\tfrac{f}{c} - \omega][1 - \chi^\varepsilon]\|_{L_4} + \|\psi^\varepsilon - z_1^a\|_{L_4} + \|z_2^a\|_{L_4}\}$$

$$\leq C[\varepsilon^{1/4} + a^{1/2} + \varepsilon^{-3/4}a^{1/2}]$$

(combining our previous results (7.14)-(7.16)). Choosing $\varepsilon = a^{1/2}$ we have for $0 < a \leq \varepsilon_0^2$ that

$$\|f - c\omega - cz^a\|_{L_4} \leq Ca^{1/8}$$

for some constant C depending only on f, c, ω and Ω.

<u>Lemma 6</u>: Suppose that z is the solution of Problem (3.3) and $\{Z_n\}$ the solution of (4.4)-(4.5). If $c(x,t) \geq 0$ and $c_t(x,t) \leq 0$ for $(x,t) \in I_T$ and z is sufficiently smooth, then

(7.17)
$$|z(\cdot,t_n) - Z_n|_{L_\infty} \leq C\Big[h^k(\|z\|_{L_\infty(W^{k,\infty})} + \|\tfrac{\partial z}{\partial t}\|_{L_2(H^k)})$$

$$+ (\Delta t)^2\Big(\Big\|\frac{\partial^3 z}{\partial t^3}\Big\|_{L_2(L_2)} + \|\tfrac{\partial z}{\partial t}\|_{L_2(L_2)} + \|z\|_{L_2(L_2)}\Big)\Big]$$

for some constant C independent of h and Δt.

Proof: Following the proof in Wheeler [15] we define for every $t \in [0,T]$, $\hat{z}(\cdot,t) \in \overset{\circ}{S}{}^k_h$ by

$$(7.18)\qquad a(\frac{\partial}{\partial x}[\hat{z} - z], \frac{\partial V}{\partial x}) + (c[\hat{z} - z], V) = 0, \quad V \in \overset{\circ}{S}{}^k_h .$$

Averaging the weak form of differential equation (3.3) at times $t = t_n$ and $t = t_{n+1}$ yields

$$(7.19)\qquad (\partial_t z_{n+1/2}, V) + a(\frac{\partial}{\partial x} z_{n+1/2}, \frac{\partial}{\partial x} V) + ([cz]_{n+1/2}, V)$$

$$= (F_{n+1/2}, V) + (\varepsilon_n, V), \quad V \in \overset{\circ}{S}{}^k_h ,$$

where

$$\varepsilon_n = \partial_t z_{n+1/2} - (z_t)_{n+1/2} = \frac{1}{2\Delta t}\int_{t_n}^{t_{n+1}} \frac{\partial^3 z}{\partial t^3}(\tau)(t_{n+1} - \tau)(t_n - \tau)d\tau .$$

Now setting $\eta = \hat{z} - z$ and $\xi_n = Z_n - \hat{z}_n$, we have for all $V \in \overset{\circ}{S}{}^k_h$ that

$$(\partial_t \xi_{n+1/2}, V) + a(\frac{\partial}{\partial x}\xi_{n+1/2}, \frac{\partial}{\partial x} V) + (c(\cdot,[n+\frac{1}{2}]\Delta t)\xi_{n+1/2}, V) = -(\varepsilon_n + \partial_t \eta_{n+1/2}, V)$$

$$+ ([c\hat{z}]_{n+1/2} - c(\cdot,[n+\frac{1}{2}]\Delta t)\hat{z}_{n+1/2}, V)$$

(using (7.18), (7.19), and (4.4)). Setting $V = \partial_t \xi_{n+1/2}$, we get

$$(7.20)\qquad \|\partial_t \xi_{n+1/2}\|_0^2 + \frac{a}{2\Delta t}[\|\frac{\partial}{\partial x}\xi_{n+1}\|_0^2 - \|\frac{\partial}{\partial x}\xi_n\|_0^2] + \|c^{1/2}(\cdot,[n+\frac{1}{2}]\Delta t)\xi_{n+1}\|_0^2$$

$$- \|c^{1/2}(\cdot,[n+\frac{1}{2}]\Delta t)\xi_n\|_0^2 \le \frac{1}{2}\|\varepsilon_n + \partial_t \eta_{n+1/2}\|_0^2 + \|\partial_t \xi_{n+1/2}\|_0^2$$

$$+ \frac{1}{2}\|[c\hat{z}]_{n+1/2} - c(\cdot,[n+\frac{1}{2}]\Delta t)\hat{z}_{n+1/2}\|_0^2 .$$

Since $c_t \le 0$,

$$\|c^{1/2}(\cdot,[n+\frac{1}{2}]\Delta t)\xi_{n+1}\|_0^2 \ge \|c^{1/2}(\cdot,[n+\frac{3}{2}]\Delta t)\xi_{n+1}\|_0^2 .$$

Making this replacement and summing inequality (7.20) from $n = 0,\ldots,J^* - 1$, we get

$$\frac{a}{2\Delta t}[\|\frac{\partial}{\partial x}\xi_{J^*}\|_0^2 - \|\frac{\partial}{\partial x}\xi_0\|_0^2] + \|c^{1/2}(\cdot,[J^*+\frac{1}{2}]\Delta t)\xi_{J^*}\|_0^2 - \|c^{1/2}(\cdot,\frac{1}{2}\Delta t)\xi_0\|_0^2$$

$$\le \frac{1}{2}\sum_{n=0}^{J^*-1}\{\|\varepsilon_n + \partial_t \eta_{n+1/2}\|_0^2 + \|[c\hat{z}]_{n+1/2} - c(\cdot,[n+\frac{1}{2}]\Delta t)\hat{z}_{n+1/2}\|_0^2\} .$$

Since $\xi_0 = 0$, we have

$$\left\|\frac{\partial}{\partial x}\xi_{J^*}\right\|_0^2 \le \frac{\Delta t}{a}\sum_{n=0}^{J^*-1}\{\|\varepsilon_n + \partial_t\eta_{n+1/2}\|_0^2 + \|[c\hat{z}]_{n+1/2} - c(\cdot,[n+\tfrac{1}{2}]\Delta t)\hat{z}_{n+1/2}\|_0^2\} .$$

Now

$$[c\hat{z}]_{n+1/2} = \left[\frac{c_{n+1}+c_n}{2}\right]\left[\frac{\hat{z}_{n+1}+\hat{z}_n}{2}\right], + \left[\frac{c_{n+1}-c_n}{2}\right]\left[\frac{\hat{z}_{n+1}-\hat{z}_n}{2}\right]$$

and so

$$\|[c\hat{z}]_{n+1/2} - c(\cdot,[n+\tfrac{1}{2}]\Delta t)\hat{z}_{n+1/2}\|_0$$

$$\le \left\|\frac{c_{n+1}+c_n}{2} - c(\cdot,[n+\tfrac{1}{2}]\Delta t)\right\|_{L_\infty}\|\hat{z}_{n+1/2}\|_0 + \left\|\frac{c_{n+1}-c_n}{2}\right\|_{L_\infty}\left\|\frac{\hat{z}_{n+1}-\hat{z}_n}{2}\right\|_0$$

$$\le \frac{(\Delta t)^2}{4}\|c_{tt}\|_{L^\infty(L^\infty)}\|\hat{z}_{n+1/2}\|_0 + \frac{(\Delta t)^2}{4}\|c_t\|_{L^\infty(L^\infty)}\|\partial_t\hat{z}_{n+1/2}\|_0 .$$

Estimating terms as in [15] we get

$$\text{(7.21)}\qquad \left\|\frac{\partial}{\partial x}\xi_{J^*}\right\|_0^2 \le \frac{2}{a}\left\{(\Delta t)^4\left\|\frac{\partial^3 z}{\partial t^3}\right\|_{L_2(L_2)} + \left\|\frac{\partial\eta}{\partial t}\right\|^2_{L_2(L_2)}\right.$$

$$+ \left(\left[\frac{\Delta t}{2}\right]^2\|c_{tt}\|_{L_\infty(L_\infty)}\right)^2\left[2\|\hat{z}\|^2_{L_2(L_2)} + (\Delta t)^2\left\|\frac{\partial\hat{z}}{\partial t}\right\|^2_{L_2(L_2)}\right]$$

$$\left. + \left(\left[\frac{\Delta t}{2}\right]^2\|c_t\|_{L_\infty(L_\infty)}\right)^2\left\|\frac{\partial\hat{z}}{\partial t}\right\|^2_{L_2(L_2)}\right\} \le C(a_*,a^*)\left\{(\Delta t)^4\left\|\frac{\partial^3 z}{\partial t^3}\right\|^2_{L_2(L_2)}\right.$$

$$\left. + \left\|\frac{\partial\eta}{\partial t}\right\|^2_{L_2(L_2)} + \|\eta\|^2_{L_2(L_2)} + (\Delta t)^4\left[\|z\|^2_{L_2(L_2)} + \left\|\frac{\partial z}{\partial t}\right\|^2_{L_2(L_2)}\right]\right\}$$

(since $\hat{z} = \eta + z$). Since by the standard Nitsche argument

$$\text{(7.22)}\qquad \|\eta(\cdot,t)\|_0 \le C(a_*,a^*)h^k\|z(\cdot,t)\|_k$$

we will get the desired result by following the argument in [15] provided

$$\left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(L_2)} \le C(a_*,a^*)h^k[\|z_t\|_{L_2(H^k)} + \|z\|_{L_2(H^k)}] .$$

Now from (7.18) we have

$$a\left(\frac{\partial}{\partial x}\eta, \frac{\partial}{\partial x}V\right) + (c\eta, V) = 0, \quad V \in \mathring{S}_h^k .$$

Hence

$$a\left(\frac{\partial}{\partial x}\eta_t, \frac{\partial}{\partial x}V\right) + (c\eta_t, V) = -(c_t\eta, V) \quad (7.23)$$

To estimate $\|\frac{\partial\eta}{\partial t}\|_{L_2(L_2)}$ we define for every $t \in [0,T]$, $\psi(\cdot,t) \in \overset{\circ}{S}{}^k_h$ by

$$a\left(\frac{\partial}{\partial x}[z_t - \psi], \frac{\partial}{\partial x}V\right) + (c[z_t - \psi], V) = 0, \quad V \in \overset{\circ}{S}{}^k_h . \quad (7.24)$$

Then from (7.23) and (7.24) we have for all $V \in \overset{\circ}{S}{}^k_h$ that

$$a\left(\frac{\partial}{\partial x}[\hat{z}_t - \psi], \frac{\partial}{\partial x}V\right) + (c[\hat{z}_t - \psi], V) = (-c_t\eta, V) .$$

Choosing $V = \hat{z}_t - \psi$ one easily gets

$$\|[\hat{z}_t - \psi](\cdot,t)\|_0 \leq C(a_*,a^*)\|\eta(\cdot,t)\|_0 .$$

Now by the standard Nitsche argument

$$\|[z_t - \psi](\cdot,t)\|_0 \leq C(a_*,a^*)h^k\|z_t(\cdot,t)\|_k . \quad (7.25)$$

Hence from (7.22) and the triangle inequality, we get

$$\|\eta_t(\cdot,t)\|_0 \leq C(a_*,a^*)h^k[\|z_t(\cdot,t)\|_k + \|z(\cdot,t)\|_k]$$

and so

$$\|\eta_t\|_{L_2(L_2)} \leq C(a_*,a^*)h^k[\|z_t\|_{L_2(H^k)} + \|z\|_{L_2(H^k)}] .$$

The remainder of the proof of the lemma is identical to the proof in [15].

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1902	2. GOVT ACCESSION NO.	3. ~~RECIPIENT'S CATALOG~~ NUMBER (9) Technical
4. TITLE (and Subtitle) (6) ERROR ESTIMATES FOR THE APPROXIMATION OF AN UNKNOWN CONSTANT COEFFICIENT IN A PARTIAL DIFFERENTIAL EQUATION.		5. ~~TYPE OF REPORT & PERIOD~~ COVERED Summary Report, ~~no specific~~ reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Richard S. Falk		8. CONTRACT OR GRANT NUMBER(s) (15) DAAG29-75-C-0024, NSF-MCS78-02737
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 7 - Numerical Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. ~~REPORT DATE~~ (11) Dec[redacted]1978
		13. ~~NUMBER OF PAGES~~ 35
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (12) 40p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(14) MRC-TSR-1902

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inverse problem

Identification problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A priori error estimates are derived for the approximation of an inverse problem in which it is desired to identify an unknown constant coefficient in a one space dimensional parabolic or two space dimensional elliptic partial differential equation whose general form is known.

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